UDC 624.07:534.1

ON THE STABILITY OF NONUNIFORMLY AGING VISCOELASTIC RODS

PMM Vol. 43, No. 4, 1979, pp. 709-721

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(Received January 10, 1979)

The stability of a nonuniformly aging viscoelastic rod is investigated. The critical force is determined in terms of parameters of the rod in the case of infinite time interval. Critical time estimates are presented for the problem covering a finite time interval.

The investigation of stability of viscoelastic rods is dealt with in a number of publications. The majority of these relate to stability of the axially compressed rod (bibliography and survey of these publications appeared in, e.g., [1–5]). There are several essentially different formulations of the problem of stability of viscoelastic rods. Thus for rods of materials with limited creep properties the problem usually concerns the determination of continuous critical loading over an infinitely long interval. If the deflection of a viscoelastic rod is limited at all times, its deformation is called stable. If, however, the deflection, as a function of time, becomes at some point unlimited, the rod is called unstable. One of the problems of stability over a specified finite time interval is the determination of the initial deflection limits whose fulfilment ensures that the deflection determined by it does not exceed a certain value. The formulated problems of stability over finite time intervals are based on the definitions of stability of motion of dynamic systems proposed by Chetaev [6]. A survey and bibliography of publications dealing with technical stability of motion are given in [7, 8].

Below, we investigate the stability of rods whose properties are defined by equations of the theory of viscoelasticity of nonuniformly aging bodies [9–12].

On the model of a nonuniformly aging viscoelastic body. The model of a nonuniformly aging viscoelastic body whose elastic and rheological properties vary with time is characterised by its specific inhomogeneity. This inhomogeneity is due to the fact that the aging process is not uniform in all elements of such bodies, which makes the age of material generally dependent on space coordinates. The variability of the age of material determines, in turn, the form of functions that define the properties of a viscoelastic body in terms of time and space coordinates.

The equations of state for such model of the viscoelastic body, which take into account the dependence of the age of material on space coordinates, can be obtained on the basis of [9]. Denoting by τ_0 the instant of stress application to an element of the viscoelastic body in the neighborhood of point $r = (x_1, x_2, x_3)$, and by $\tau_1^* = \tau_1^*(r)$ the instant of production generation of that element. The instant of observation is denoted by t (absolute time), with the reference point for time selected arbitrarily. Let $\tau_1 = \tau_1(r)$ be the age of a material element in the neighborhood of point $r = (x_1, x_2, x_3)$ of the considered body at instant τ_0 of stress application. We obviously have

$$\tau_1(r) = \tau_0 - \tau_1^*(r) \tag{0.1}$$

The equation of state that links the strain $\varepsilon_x(t)$ and the stress $\sigma_x(t)$ in a nonuniformly aging viscoelastic body subjected to uniaxial stress is, then, of the form

$$\varepsilon_x(t) = \frac{\sigma_x(t)}{E(t-\tau_1^*)} - \int_{\tau_0}^{t} \sigma_x(\tau) K(t-\tau_1^*, \tau-\tau_1^*)\, d\tau \tag{0.2}$$

$$K(t, \tau) = \frac{\partial}{\partial\tau} \Delta(t, \tau) = \frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + N(t, \tau)\right] \tag{0.3}$$

where $K(t, \tau)$ is the creep kernel for such body, i.e. when $\tau_1^*(x) \equiv 0$, $E(t)$ is the variable modulus of instantaneous elastic strain, and $N(t, \tau)$ is the measure of creep of the material.

We substitute in Eq. (0.2) the age $\tau_1(r)$ of a material element at instant τ_0 of stress application for the age $\tau_1^*(r)$ at the instant of its production (generation). For simplicity we assume that in a uniaxial stress state the age τ_1 of the material depends only on one coordinate x. We select for definiteness the instant of production (generation) of the viscoelastic body element at the coordinate $x = 0$ as the time reference point, and introduce the age $\rho(x)$ of the element. (Function $\rho(x)$ of nonuniform aging defines the dependence of the age of material of an aging viscoelastic body on coordinates).

Taking the aforesaid and (0.1) into account we represent the equation of state in the form

$$\varepsilon_x(t) = \frac{\sigma_x(t)}{E[t+\rho(x)]} - \int_{\tau_0}^{t} \sigma_x(\tau) K[t+\rho(x), \tau+\rho(x)]\, d\tau \tag{0.4}$$

If the instants of stress application to various elements of the viscoelastic body are different, i.e. $\tau_0 = \tau_0(x)$, the equation of state is of the form

$$\varepsilon_x(t) = \frac{\sigma_x(t)}{E[t-\tau_1^*(x)]} - \int_{\tau_0(x)}^{t} \sigma_x(\tau) K[t-\tau_1^*(x), \tau-\tau_1^*(x)]\, d\tau \tag{0.5}$$

Function $\tau_1^* = \tau_1^*(x)$ represents the instant of generation of the element at coordinate x, and function $\tau_0 = \tau_0(x)$ defines the instant of stress application to that element. Obviously $\tau_1^*(x) \leqslant \tau_0(x)$.

The equations of state (0.4) and (0.5) are the determinining relations for the creep of nonuniformly aging bodies in a uniaxial state of stress when the strains do not exceed the proportionality limit. The equations of state for the general case of three-dimensional state of stress with strains, although small but exceeding the proportionality limit, appear in [9–11].

For a constant modulus of instantaneous elastic strain the kernel is of the form $K(t, \tau) = \partial N(t, \tau) / \partial\tau$. According to [12] the measure of creep $N(t, \tau)$ can be defined by the formula

$$N(t, \tau) = \varphi(\tau)\, f(t-\tau) \tag{0.6}$$

Function $\varphi(\tau)$ determines the aging process of the rod material. Expanding function

$f(t-\tau)$ in an exponential series and retaining in it only the first two terms, we obtain

$$N(t, \tau) = \varphi(\tau)[1 - e^{-\gamma(t-\tau)}] \tag{0.7}$$

Equation (0.4 with conditions (0.6) and (0.7) constitute the basis of subsequent investigation.

For each fixed x and given strain formula (0.5) is the Volterra integral equation of the second kind with respect to stresses (a survey of publications on these equations appears in [13]). If for a fixed x functions $K(t-\tau_1^*(x), \tau-\tau_1^*(x))E(t-\tau_1^*(x))$ and $\varepsilon_x(t)E(t-\tau_1^*(x))$ are square integrable when $\tau_0(x) \leqslant t, \tau \leqslant T$, then Eq. (0.5) has the unique solution $\sigma_x(t)$, $\tau_0(x) \leqslant t \leqslant T$ which is square integrable.

1. Stability over an infinite time interval. Let us consider a nonuniformly aging viscoelastic rod whose length can be assumed, without loss of generality, to be equal unity. The undeformed rod lies on the Ox-axis. At instant t_0 the longitudinal force P is applied to it. We denote the rod deflection measured from the compressive force line of action at point x at instant of time $t \geqslant t_0$. It is assumed that at the instant of time $t_0 - 0$ immediately preceding the application of force the rod had an initial deflection $y_0(x)$, i.e.

$$y(t_0 - 0, x) = y_0(x) \tag{1.1}$$

where functions $y_0(x)$ is given and has two continuous derivatives with respect to $x \in [0, 1]$.

We call the rod stable, if its deflection defined in terms of t and x is bounded, i.e.

$$\sup_{t \geqslant t_0,\; x \in [0,1]} |y(t, x)| < \infty \tag{1.2}$$

We further assume that the longitudinal strain distribution over the rod section conforms to the law of plane sections. By virtue of (0.4) the equation for the deflection $y(t, x)$ of the viscoelastic rod of nonuniformly aging material is of the form

$$\frac{\partial^2 y(t, x)}{\partial x^2} + \frac{P}{I}\left\{\frac{y(t, x)}{E} - \int_{t_0}^{t} y(\tau, x)\frac{\partial}{\partial \tau}[\varphi(\tau + \rho(x))(1 - e^{-\gamma(t-\tau)})]\,d\tau\right\} = \frac{d^2 y_0(x)}{dx^2} \tag{1.3}$$

where the positive bounded continuous function $\varphi(\tau)$ approaches, as $\tau \to \infty$, some constant $C_0 > 0$ which represents the measure of creep limit of the material in its old age; function $\rho(x)$ is piecewise-continuous and bounded; the specified constant $\gamma > 0$; E is the modulus of instantaneous elastic strain, and I is the moment of inertia of the rod about the longitudinal axis.

The boundary conditions for Eq. (1.3) are defined by

$$y(t, 0)\cos\alpha + \frac{\partial y(t, 0)}{\partial x}\sin\alpha = 0 \tag{1.4}$$

$$y(t, 1)\cos\beta + \frac{\partial y(t, 1)}{\partial x}\sin\beta = 0$$

The numbers α and β are specified so that in the elastic problem the differential equation for deflections (i.e. Eq. (1.3) for $t = t_0$) measured from the force action line, is of second order. Boundary conditions for specific types of rod fixing are obtained by a suitable selection of parameters α and β (e.g., $\alpha = \beta = 0$ for hinged rod ends, $\alpha = 0, \beta = \pi / 2$ for a rod with one end rigidly fixed and the other free; other possible boundary forms that conform to given constraint types can be found in [14]).

Let us transform Eq. (1.3) assuming that function $\partial^4 y(t, x) / \partial x^2 \partial t^2$ exists and is continuous. Differentiating (1.3) twice with respect to t, we obtain

$$\frac{\partial^4 y(t, x)}{\partial x^2 \partial t^2} + \gamma \frac{\partial^3 y(t, x)}{\partial x^2 \partial t} + \frac{P}{IE} \frac{\partial^2 y(t, x)}{\partial t^2} + \frac{P\gamma}{IE} \frac{\partial y(t, x)}{\partial t} [1 + E\varphi(t + \rho(x))] = 0 \qquad (t \geqslant t_0,\ 0 \leqslant x \leqslant 1) \tag{1.5}$$

Boundary conditions for Eq. (1.5) are of the form (1.4).

One of the initial conditions for $t = t_0$ for this equation is obtained by setting in (1.3) $t = t_0$. We obtain

$$\frac{\partial^2 y(t_0, x)}{\partial x^2} = -\frac{P}{IE} y(t_0, x) + \frac{d^2 y_0(x)}{dx^2} \tag{1.6}$$

The second initial condition is obtained by differentiating both parts of (1.3) with respect to t and making t approach t_0, which yields

$$\frac{\partial^3 y(t_0, x)}{\partial x^2 \partial t} + \frac{P}{IE} \frac{\partial y(t_0, x)}{\partial t} = -\frac{P\gamma}{I} y(t_0, x)\, \varphi(t_0 + \rho(x)) \tag{1.7}$$

The conditions of stability are obtained by deriving the solution of problem (1.4)–(1.7), (1.1) in the form of series expansion in eigenvalues of the elastic problem. We denote by λ_n the eigenvalues of the elastic problem and by $\psi_n(x)$ the related sequence of eigenfunctions. Functions $\psi_n(x)$ satisfy boundary conditions (1.4) throughout which ψ_n is to be substituted for y and, also, the equations

$$\frac{d^2 \psi_n(x)}{dx^2} + \lambda_n \psi_n(x) = 0 \qquad (0 \leqslant x \leqslant 1) \tag{1.8}$$

We recall the existence of an infinitely increasing sequence of real eigenvalues λ_n and of the related orthonormal set of eigenfunctions ψ_n (see [15]), with the deflection $y(t, x)$ expanding for any $t \geqslant t_0$ in a uniformly convergent Fourier series in $x \in [0, 1]$ of the form

$$y(t, x) = \sum_{n=0}^{\infty} A_n(t) \psi_n(x), \quad A_n(t) = \int_0^1 y(t, x) \psi_n(x)\, dx \tag{1.9}$$

On the assumptions made above about the continuity of derivative $\partial^4 y(t, x) / \partial x^2 \partial t^2$ series (1.9) may be differentiated twice with respect to t and to x under the summation sign. To prove this we first point out that when

$$\frac{\partial^2 y(t, x)}{\partial t^2} = \sum_{n=0}^{\infty} B_n(t) \psi_n(x), \quad B_n(t) = \int_0^1 \frac{\partial^2 y(t, x)}{\partial t^2} \psi_n(x)\, dx$$

then, taking into account (1.9), we have the coefficients $B_n(t) = A_n^{\cdot\cdot}$ (dots denote differentiation with respect to t). Let furthermore

$$\frac{\partial^4 y(t,x)}{\partial t^2 \partial x^2} = \sum_{n=0}^{\infty} Q_n(t)\,\psi_n(x)$$

Then integrating twice by parts we obtain

$$Q_n(t) = \int_0^1 \frac{\partial^4 y(t,y)}{\partial t^2 \partial x^2}\,\psi_n(x)\,dx = \psi_n(1)\frac{\partial y(t,1)}{\partial x} - \psi_n(0)\frac{\partial y(t,0)}{\partial x} +$$

$$\frac{d\psi_n(0)}{dx}\,y(t,0) - \frac{d\psi_n(1)}{dx}\,y(t,1) + I_1(t),\quad I_1(t) = \int_0^1 \frac{d^2\psi_n(x)}{dx^2}\,\frac{\partial^2 y(t,x)}{\partial t^2}\,dx$$

However on the strength of (1.8)

$$I_1(t) = \lambda_n \int_0^1 \psi_n(x)\frac{\partial^2 y(t,x)}{\partial t^2}\,dx = -\lambda_n B_n(t) = -\lambda_n A_n^{\cdot\cdot}(t)$$

Moreover, from the first of boundary conditions (1.4) we obtain for y and ψ_n a system of equation in $\cos\alpha$ and $\sin\alpha$, which for any α has a nontrivial solution, which means that its determinant is zero, i.e.

$$y(t,0)\frac{d\psi_n(0)}{dx} - \psi_n(0)\frac{\partial y(t,0)}{dx} = 0$$

Similarly

$$\psi_n(1)\frac{\partial y(t,1)}{\partial x} - \frac{d\psi_n(1)}{\partial x}\,y(t,1) = 0$$

Hence we conclude that

$$\frac{\partial^4 y(t,x)}{\partial t^2 \partial x^2} = -\sum_{n=0}^{\infty}\lambda_n B_n(t)\,\psi_n(x) = \sum_{n=0}^{\infty} A_n^{\cdot\cdot}(t)\frac{d^2\psi_n(x)}{dx^2}$$

The feasibility of differentiating the expression under the summation sign is proved.

Let us write the equation which determine coefficients $A_n(t)$ in expansion (1.9). For this we substitute series (1.9) into (1.5). Taking into account (1.8) and thē orthonormality of functions $\psi_n(x)$, we obtain

$$[A_n^{\cdot\cdot}(t) + \gamma A_n^{\cdot}(t)]\,\mu_n^{-1} + \gamma E \sum_{m=0}^{\infty} A_m^{\cdot}(t)\,\beta_{mn}(t) = 0 \tag{1.10}$$

$$\mu_n = \left[1 - \frac{IE}{P}\lambda_n\right]^{-1},\quad t \geqslant t_0$$

$$\beta_{mn}(t) = \int_0^1 \psi_n(x)\,\psi_m(x)\,\varphi(t + \rho(x))\,dx$$

Initial conditions for Eq. (1.10) are obtained by substituting (1.9) into (1.6) and (1.7). As a result we have

$$A_n(t_0) = -A_n^{\circ}\lambda_n\mu_n IEP^{-1} \tag{1.11}$$

$$A_n^{\cdot}(t_0) = -\gamma\mu_n E\sum_{m=0}^{\infty} A_m(t_0)\,\beta_{mn}(t_0)$$

$$A_n^\circ = \int_0^1 y_0(x)\,\psi_n(x)\,dx$$

where $\overset{\circ}{A}_n$ is the Fourier coefficient of function $y_0(x)$.

In stability investigations it is reasonable to assume that force P is smaller than the Euler's critical force, i.e.

$$P < IE\lambda_0 \tag{1.12}$$

where λ_0 is the minimal eigenvalue of the elastic problem. In what follows we assume that condition (1.12) is satisfied.

On the strength of the Parceval equality (see [16]) the formula

$$u(t) = \sum_{n=0}^{\infty} A_n^2(t) = \int_0^1 y^2(t,x)\,dx \tag{1.13}$$

is valid, which, in particular, means that the solution $A_n(t)$ of the Cauchy problem (1.10), (1.11) can be considered to be an element of the coordinate Hilbert space l_2 for any t. Note that the existence and uniqueness of solution of the Cauchy problem (1.10), (1.11) was proved in [11].

Using the second Liapunov method we shall now prove that the series $u(t)$ is bounded for all $t \geqslant t_0$. We introduce in the analysis the scalar function $V(t)$

$$V(t) = \sum_{n=0}^{\infty} A_n^{\cdot 2}(t)$$

whose derivative we shall determine along the trajectory of system (1.10). We have

$$V^{\cdot}(t) = -2\gamma V(t) - \tag{1.14}$$

$$2\gamma E \sum_{n=0}^{\infty} A_n^{\cdot}(t)\,\mu_n \sum_{m=0}^{\infty} A_m^{\cdot}(t)\,\beta_{mn}(t)$$

We transform the right-hand side of formula (1.14), and represent $\beta_{mn}(t)$ in the form

$$\beta_{mn}(t) = C_0\delta_{mn} + \bar{\beta}_{mn} \tag{1.15}$$

$$\bar{\beta}_{mn}(t) = \int_0^1 \psi_n(x)\,\psi_m(x)\,(\varphi(t+\rho(x)) - C_0)\,dx$$

where δ_{mn} is the Kronecker delta.

On the strength of assumptions about functions φ and ρ the equality

$$\lim_{t\to\infty} \bar{\beta}_{mn}(t) = 0 \tag{1.16}$$

is uniformly valid with respect to m and n.

Taking into account (1.15) we obtain

$$2\gamma E \sum_{n=0}^{\infty} A_n^{\cdot}\mu_n \sum_{m=0}^{\infty} A_m^{\cdot}(t)\,\beta_{mn}(t) = \tag{1.17}$$

$$2\gamma E C_0 \sum_{n=0}^{\infty} A_n^{\cdot 2}(t)\,\mu_n + 2\gamma E \Sigma_{nm}$$

$$\Sigma_{nm} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n^{\cdot}(t) A_m^{\cdot}(t) \mu_n \beta_{mn}(t)$$

Then, on the strength of (1.12), we have the inequality

$$\sum_{n=0}^{\infty} A_n^{\cdot 2}(t) \mu_n > \mu_0 V(t) \tag{1.18}$$

and on the strength of the Cauchy–Buniakowski inequality we have

$$\Sigma_{nm} \leqslant [V(t)]^{1/2} \Big[\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} A_m^{\cdot}(t) \mu_n \bar{\beta}_{mn}(t) \Big)^2 \Big]^{1/2} \leqslant \tag{1.19}$$

$$V(t) \Big[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mu_n^2 \bar{\beta}_{mn}^2(t) \Big]^{1/2} =$$

$$V(t) \Big[\sum_{n=0}^{\infty} \mu_n^2 \int_0^1 \psi_n^2(x) (\varphi(t+\rho(x)) - C_0)^2 \, dx \Big]^{1/2}$$

In the last transformation when deriving formula (1.19) we used the Parceval equality in the form

$$\sum_{n=0}^{\infty} \bar{\beta}_{mn}^2(t) = \int_0^1 \psi_n^2(x) (\varphi(t+\rho(x)) - C_0)^2 \, dx$$

However for large n the asymptotic formula

$$\lambda_n = n + O\left(\frac{1}{n}\right) \tag{1.20}$$

is valid (see [15]). Since in conformity with (1.12) all quantities μ_n^2 are positive, it follows from (1.20) and (1.12) that the series

$$\sum_{n=0}^{\infty} \mu_n^2 = C_1 < \infty \tag{1.21}$$

is convergent. The letters C_i $(i = 1, 2, \ldots)$ denote in it some positive constants. From estimate (1.21) and the normalized properties of eigenfunctions ψ_n follows the identity

$$\sum_{n=0}^{\infty} \mu_n^2 \Big[\int_0^1 \psi_n^2(x) [\varphi(t+\rho(x)) - C_0]^2 \, dx \Big]^{1/2} \leqslant C_1 \varphi_1(t) \tag{1.22}$$

$$\varphi_1(t) = \max_{0 \leqslant x \leqslant 1} (\varphi(t+\rho(x)) - C_0)$$

where C_1 is defined by formula (1.21).

The substitution of expressions (1.17), (1.19), and (1.22) into the right-hand side of (1.14) yields

$$V^{\cdot}(t) \leqslant \{-2\gamma [1 + EC_0\mu_0] + 2\gamma EC_1\} \varphi_1(t) V(t) \tag{1.23}$$

Let us now require the expression in brackets in the right-hand side of (1.23) to be positive. This is achieved when the compressive force P satisfies the condition (*)

*) This was pointed out by A. S. Lozovskii in a particular case.

$$P < EI\lambda_0 (1 + EC_0)^{-1} \tag{1.24}$$

We call critical the compressive force defined by the right-hand side of (1.24) under conditions of continuous loading.

We shall now establish the boundedness of function $V(t)$ when $t \geqslant t_0$ and condition (1.24) is satisfied. Note that by virtue of assumptions made about functions φ and ρ we have $\lim_{t\to\infty} \varphi_1(t) = 0$. This means that there exists a t_1 such that for all $t \geqslant t_1$ and some $\varepsilon > 0$ the inequality

$$-[1 + EC_0 \mu_0] + E\varphi_1(t) \leqslant -\varepsilon$$

is valid.

It is also possible to show that

$$u(t_0) < \infty, \quad V(t_0) < \infty \tag{1.25}$$

Indeed, by virtue of (1.11) and the Parceval equality

$$u(t_0) \leqslant C \sum_{n=0}^{\infty} (A_n^\circ)^2 = C \int_0^1 y_0^2(x)\, dx < \infty$$

Similarly, using the Cauchy–Buniakowski inequality, the Parceval equality, and formulas (1.11) and (1.20) we establish that

$$V(t_0) \leqslant \gamma^2 E^2 \sum_{n=0}^{\infty} \mu_n^2 \sum_{m=0}^{\infty} \left(A_m^\circ\right)^2 \sum_{m=0}^{\infty} \beta_{mn}^2(t_0) = \tag{1.26}$$

$$\gamma^2 E^2 \sum_{n=0}^{\infty} \mu_n^2 \int_0^1 y_0^2(x) dx \int_0^1 \psi_n^2(z)\, \varphi^2(t_0 + \rho(z))\, dz \leqslant$$

$$C_2 \int_0^1 y_0^2(x)\, dx < \infty$$

$$C_2 = \gamma^2 E^2 \sum_{n=0}^{\infty} \mu_n^2 \max_x \varphi^2(t_0 + \rho(x))$$

Inequalities (1.23) and (1.25) imply the unboundedness of function $V(t)$ in the interval $[t_0, t_1]$. Hence taking into account (1.23) and (1.24) we conclude that a constant $\varkappa > 0$ can be found for which

$$V(t) \leqslant \varkappa e^{-\varkappa(t-t_0)}, \quad t \geqslant t_0 \tag{1.27}$$

From inequality (1.27) and the definition of function $V(t)$ on the strength of (1.25) we have

$$\sup_{t \geqslant t_0} u(t) < \infty$$

This, with the Parceval equality (1.13), implies that

$$\sup_{t \geqslant t_0} \int_0^1 y^2(t, x)\, dx < \infty \tag{1.28}$$

Let us now show that (1.2), i.e. the stability of the viscoelastic rod over an infinite time interval, follows from (1.28). We denote by $G(x, \xi)$ the elastic problem Green's function, i.e. the Green's function of Eq. (1.3) for $\varphi = y_0 = 0$ with boundary conditions (1.4). By virtue of (1.3) the following representation is valid:

$$y(t,x)=\int_0^1 G(x,\xi)\left\{\frac{d^2y_0(\xi)}{d\xi^2}+\right. \tag{1.29}$$

$$\left.\frac{P}{I}\int_{t_0}^{t} y(\tau,\xi)\frac{\partial}{\partial\tau}[\varphi(\tau+\rho(\xi))(1-e^{-\gamma(t-\tau)})]\,d\tau\right\}d\xi$$

Integration of this equality by parts reduces it to the form

$$y(t,x)=\int_0^1 G(x,\xi)\left[\frac{d^2y_0(\xi)}{d\xi^2}+\right. \tag{1.30}$$

$$\left.\frac{P}{I}y(t_0,\xi)\varphi(t_0+\rho(\xi))(1-e^{-\gamma(t-t_0)})\right]d\xi-$$

$$\frac{P}{I}\int_0^1 G(x,\xi)\int_{t_0}^{t}\frac{\partial y(\tau,\xi)}{\partial\tau}[\varphi(\tau+\rho(\xi))(1-e^{-\gamma(t-\tau)})]\,d\tau\,d\xi$$

in which the first term in the right-hand side is obviously bounded for all $t\geqslant t_0$. Let us estimate the second [term]. Taking into account the Cauchy–Buniakowski inequality, the Parceval equality, and formula (1.27) we obtain

$$\left|\frac{P}{I}\int_0^1 G(x,\xi)\,d\xi\int_{t_0}^{t}\frac{\partial y(\tau,\xi)}{\partial\tau}[\varphi(\tau+\rho(\xi))\times\right. \tag{1.31}$$

$$\left.(1-e^{-\gamma(t-\tau)})\right]d\tau\right|\leqslant\int_{t_0}^{t}C_3(\tau)\int_0^1\left|\frac{\partial y(\tau,\xi)}{\partial\tau}\right|d\xi\,d\tau\leqslant$$

$$\int_{t_0}^{t}C_3(\tau)\,d\tau\left(\int_0^1\left(\frac{\partial y(\tau,\xi)}{\partial\tau}\right)^2d\xi\right)^{1/2}=\int_{t_0}^{t}C_3(\tau)V(\tau)\,d\tau$$

$$C_3(t)=\frac{P}{I}\max_{x,\,\xi}|G(x,\xi)\varphi(t+\rho(\xi))|$$

from which and (1.30) follows (1.2).

The critical value of the compressive force P in the case considered here is, thus, defined by the right-hand side of formula (1.24). It is independent of function $\rho(x)$ which defines the inhomogeneity of the aging material of the viscoelastic rod. Note that the critical value of the compressive force in the case of homogeneous material of such rod was obtained in [18].

Remark 1. Inequality (1.24) represents only the sufficient condition of stability, but by no means the necessary one, since the critical force that satisfies (1.12) can exceed the [one defined by the] right-hand side of (1.24) over any finite time interval, without impairing the rod stability. To prove this it is sufficients to point out that, as implied by (1.23) and (1.25), the [rod] deflection over any finite time interval $[t_0,t_1]$ is limited. Hence, if estimate (1.24) holds for $t\geqslant t_1$, the inequality (1.27) from which (1.2) is derived must also hold.

Remark 2. Let us show that when

$$IE\lambda_0(1+EC_0)^{-1} < P < E\lambda_0 I \tag{1.32}$$

the rod is unstable, i.e. inequality (1.2) is violated. To prove this we write Eq. (1.10) for $n = 0$. From formula (1.15) follows that

$$(A_0^{\cdot\cdot} + \gamma A_0^{\cdot})(1 - IE\lambda_0 P^{-1}) + \gamma E C_0 A_0^{\cdot} + \gamma E \sum_{m=0}^{\infty} A_m^{\cdot}(t)\,\bar{\beta}_{m0}(t) = 0 \tag{1.33}$$

Let us assume that the opposite, i.e. that (1.2) is valid. Then from (1.29), (1.2), and the Parceval equality we can derive

$$\sup_{t \geqslant t_0} V(t) < \infty$$

This together with (1.33), (1.32), and (1.16) indicates the existence of an initial deflection $y_0(x)$ such that the respective coefficient $A_0(t)$ infinitely increases. But by virtue of the definition of function $A_0(t)$ this contradicts (1.2). This proves the rod instability when condition (1.32) is violated.

2. Stability over a finite time interval. 1°. First, we shall investigate the [problem of] rod stability over a finite time interval in the following formulation. A finite time interval $[t_0, T]$ and the number $y^* > 0$ are specified. We have to determine the constraints on the initial deflection $y_0(x)$ whose fulfilment results in

$$\sup_t \sup_x |y(t, x)| \leqslant y^*, \quad t_0 \leqslant t \leqslant T, \quad 0 \leqslant x \leqslant 1 \tag{2.1}$$

The compressive force P is assumed to satisfy the inequality (1.12).

To determine constraints on the initial deflection we use formula (1.30) where we evaluate its separate terms. By virtue of (1.29)

$$y(t_0, x) = \int_0^1 G(x, \xi)\frac{d^2 y_0(\xi)}{d\xi^2}\,d\xi$$

The substitution of this equality into (1.30) shows that the first integral in the right-hand side of (1.30) does not exceed $\psi_1(t)$ defined by

$$\psi_1(t) = C_4(t)\int_0^1 \left|\frac{d^2 y_0(\xi)}{d\xi^2}\right| d\xi \tag{2.2}$$

$$C_4(t) = \max_{x, \xi, \xi_1}\left[|G(x,\xi)| + \frac{P}{I}\,|G(x,\xi)\,G(\xi,\xi_1)\,\varphi(t_0 + \rho(\xi))\,(1 - e^{-\gamma(t-t_0)})|\right]$$

$$0 \leqslant x, \xi, \xi_1 \leqslant 1$$

To evaluate the second integral in (1.30) we use formula (1.31) in which we define more exactly the upper bound of function $V(t)$, $t_0 \leqslant t \leqslant T$. Taking into account (1.23) and (1.26), we conclude that

$$V(t) \leqslant C_2 \int_0^1 y_0^2(x)\,dx\,C_6(t) \tag{2.3}$$

$$C_6(t) = \exp\int_{t_0}^{t} C_5(s)\,ds$$

where $C_5(t)$ is equal to the expression in braces in (1.23). From (2.4) and (1.31) we obtain the estimate of the second integral in (1.30). This estimate together with (2.2) finally shows that the maximum deflection for $t_0 \leqslant t \leqslant T$ and $0 \leqslant x \leqslant 1$ (i.e. the left-hand side of (2.1)) does not exceed I_2 defined by

$$I_2 = C_4(T) \int_0^1 \left| \frac{d^2 y_0(x)}{dx^2} \right| dx + \int_{t_0}^{T} C_3(\tau) \left[C_2 \int_0^1 y_0^2(x)\, dx C_6(\tau) \right]^{1/2} d\tau \qquad (2.4)$$

This means that for any deflection $y_0(x)$ that satisfies the condition $I_2 \leqslant y^*$, condition (2.1) is satisfied, i.e. the rod is stable. Note that the derived stability condition depends on the initial deflection and its second derivative in an integral way only.

2°. Let us consider the rod stability [problem] in the following formulation. The initial deflection $y_0(x)$ and the number $y^* > 0$ are specified. We have to determine the first instant of time $t = t_1$ at which the maximum deflection is equal to the critical value y^*, i.e.

$$\max_x |y(t, x)| = y^*, \quad 0 \leqslant x \leqslant 1 \qquad (2.5)$$

Since an exact determination of instant t_1 with the use of condition (2.5) is only possible in exceptional cases, hence various estimates of t_1 are of interest.

Let us determine the lower bound t_1^- of instant t_1 and, by the same token, evaluate the interval $[t_0, t_1^-]$ in which the deflection does not exceed y^*.

We assume that at $t = t_0$ the left-hand side of (2.5) does not exceed y^*. To obtain the required estimate we use equality (1.30). The estimate of the first integral in (1.30) is derived from formula (2.2).

The second integral in (1.30) is, in conformity with (2.3) and (1.31) by the function $\psi_2(t)$

$$\psi_2(t) = C_2 \int_0^1 y_0^2(x)\, dx \int_{t_0}^{t} C_3(s) C_6(s)\, ds \geqslant \int_{t_0}^{t} C_3(s) V(s)\, ds$$

We, thus, finally obtain

$$\sup_x |y(t, x)| \leqslant \psi_1(t) + \psi_2(t) \qquad (2.6)$$

where the right-hand side is a monotonically increasing function of t. If we denote the root of equation $\psi_1(t) + \psi_2(t) = y^*$ by t_1^-, then by virtue of (2.6) we have $t_1 \geqslant t_1^-$.

Note that for specific types of rod end fixing the instant t_1 can be estimated using the method expounded here also in the case of more general functions defining the measure of creep.

Let us illustrate the proposed method with the example of the problem of stability of a nonuniformly aging viscoelastic rod with one end fixed and the other free.

The equation for the deflection $y(t, x)$ is of the form

$$\frac{d^2 y_0(x)}{dx^2} = \frac{\partial^2 y(t, x)}{\partial x^2} + \frac{P}{I}\left\{ y(t, x) - \int_{t_0}^{t} y(\tau, x) \frac{\partial}{\partial \tau} N(t + \rho(x), \tau + \rho(x))\, d\tau \right\} \qquad (2.7)$$

$$y(t, 0) = 0, \qquad \frac{\partial y(t, 1)}{\partial x} = 0$$

where the measure of creep $N(t, \tau)$ is bounded, monotonically increasing with t for any fixed τ, monotonically decreasing with τ for fixed t and $N(t, \tau) = 0$.

We assume that the second derivative of the initial deflection $y_0(x)$ is nonpositive. From this and (2.7) follows that the deflection $y(t, x)$ is convex with respect to x for any fixed t. Hence, owing to the boundary conditions (2.7), the maximum deflection $y(t, x)$ occurs at $x = 1$.

Using Green's function of the elastic problem and taking into account (2.7), we obtain for $y(t, x)$ the expression

$$y(t, 1) = \int_0^1 K(s) f(t, s)\, ds \tag{2.8}$$

$$K(s) = -\frac{1}{\omega} \operatorname{tg} \omega \cos \omega (1-s) + \frac{1}{\omega} \sin \omega (1-s)$$

$$\omega^2 = P/I$$

$$f(t, s) = \int_{t_0}^{t} y(\tau, s) \frac{\partial}{\partial t} N(t + \rho(s), \tau + \rho(s))\, d\tau + \frac{d^2 y_0(s)}{ds^2}$$

By virtue of (2.8) the following inequality is valid:

$$y(t, 1) \leqslant \int_{t_0}^{t} y(\tau, 1) K_1(t, \tau)\, d\tau + K_2 \tag{2.9}$$

$$K_1(t, \tau) = \int_0^1 \left| \frac{\partial}{\partial \tau} N(t + \rho(s), \tau + \rho(s)) K(s) \right| ds \tag{2.10}$$

$$K_2 = \int_0^1 \left| \frac{d^2 y_0(s)}{ds^2} K(s) \right| ds$$

Let us now construct function $\psi_3(t)$ in the form of the sum of series

$$\psi_3(t) = K_2 \left(1 + \int_{t_0}^{t} K_1(t, \tau)\, d\tau + \int_{t_0}^{t} K_1(t, \tau) \int_{\tau_0}^{\tau} K_1(\tau, \tau_1)\, d\tau_1 + \ldots\right) \tag{2.11}$$

Note that according to (2.10) series (2.11) is absolutely and uniformly convergent. Integration of the right-hand side of (2.9) yields $y(t, 1) \leqslant \psi_3(t)$. Hence the root t_1 of equation $\psi_3(t) = y^*$ represents the lower bound of the critical time $t_1^- \leqslant t_1$.

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Translated by J. J. D.